Taylor expansion remainder using integration by parts

The method of integration by parts, Stewart pages 511ff, looks at first as though it were chiefly a method of taking antiderivatives. Actually, its use in proofs is just as important. In the present file the method is used to prove the Taylor expansion with remainder. This way is different from and perhaps easier than the way Stewart does it on his pages 798ff. In what follows, the reader is to assume that the derivatives and higher derivatives exist and are continuous on the domains where they are needed. Let us begin with a form of the fundamental law of calculus:

$$f(x) - f(a) = \int_a^x f'(t)dt.$$

Multiplying by 1 can do no harm:

$$f(x) - f(a) = \int_a^x f'(t)\,1\,dt.$$

It is true that $(x-t)^0 = 1$ except where t is exactly equal to x, so

$$f(x) = f(a) + \int_a^x f'(t)(x-t)^0 dt.$$

Now let us do integration by parts using $u = f'(t)$ and $dv = (x-t)^0 dt$. We get

$$f(x) = f(a) + \left[-f'(t)\frac{(x-t)^1}{1} \right]_{t=a}^{t=x} + \int_a^x f''(t)\frac{(x-t)^1}{1} dt$$

and hence

$$f(x) = f(a) - 0 + f'(a)\frac{(x-a)^1}{1} + \int_a^x f''(t)\frac{(x-t)^1}{1} dt.$$

Again we do integration by parts using $u = f''(t)$ and $dv = \dfrac{(x-t)^1}{1} dt$. We get

$$f(x) = f(a) + f'(a)\frac{(x-a)^1}{1} + \left[-f''(t)\frac{(x-t)^2}{2\cdot 1}\right]_{t=a}^{t=x} + \int_a^x f'''(t)\frac{(x-t)^2}{2\cdot 1}dt$$

and hence

$$f(x) = f(a) + f'(a)\frac{(x-a)^1}{1} - 0 + f''(a)\frac{(x-a)^2}{2\cdot 1} + \int_a^x f'''(t)\frac{(x-t)^2}{2\cdot 1}dt.$$

One more integration by parts will help the intuition, using $u = f'''(t)$ and $dv = \frac{(x-t)^2}{2\cdot 1}dt$. Writing $f^{(4)}$ to mean f'''' we get

$$f(x) = f(a) + f'(a)\frac{(x-a)^1}{1} + f''(a)\frac{(x-a)^2}{2\cdot 1} + \left[-f'''(t)\frac{(x-t)^3}{3\cdot 2\cdot 1}\right]_{t=a}^{t=x} + \int_a^x f^{(4)}(t)\frac{(x-t)^3}{3\cdot 2\cdot 1}dt$$

and hence

$$f(x) = f(a) + f'(a)\frac{(x-a)^1}{1} + f''(a)\frac{(x-a)^2}{2\cdot 1} - 0 + f'''(a)\frac{(x-a)^3}{3\cdot 2\cdot 1} + \int_a^x f^{(4)}(t)\frac{(x-t)^3}{3\cdot 2\cdot 1}dt.$$

It is usual to write those denominators with the ! sign for factorial. Doing that and using a big sigma sign for addition, we get finally

$$f(x) = \sum_{i=0}^{n} f^{(i)}(a)\frac{(x-a)^i}{i!} + \int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}dt.$$

(Yes, the factorial of zero is equal to one.)

Thus, $f(x)$ equals its Taylor polynomial of degree n plus an error term represented by the integral. If we approximate $f(x)$ by its Taylor polynomial, the error is the integral we ignored. We can estimate this error just as we did for Continuously Differentiable Functions.